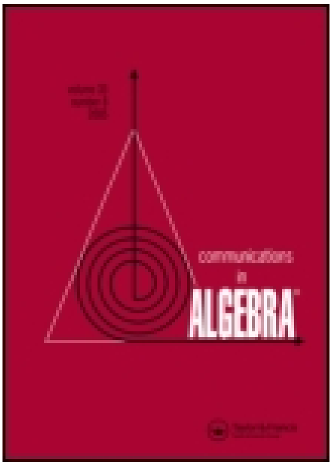

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lagb20

On Cosets of the Unit Loop of Octonion Integers

Mohammed Abouzaid [a], Jarod Alper [b], Steve DiMauro [c], Justin Grosslight [d] & Derek Smith [e]

[a] Mathematics Department, University of Chicago, Chicago, Illinois, USA

[b] Mathematics Department, Stanford University, Stanford, California, USA

[c] Silver Spring, Maryland, USA

[d] History of Science Department, Harvard University, Cambridge, Massachusetts, USA

[e] Mathematics Department, Lafayette College, Easton, Pennsylvania, USA

Published online: 07 Feb 2007.

Communications in Algebra®, 35: 207–214, 2007

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870601041540

ON COSETS OF THE UNIT LOOP OF OCTONION INTEGERS

Mohammed Abouzaid
Mathematics Department, University of Chicago, Chicago, Illinois, USA

Jarod Alper
Mathematics Department, Stanford University, Stanford, California, USA

Steve DiMauro
Silver Spring, Maryland, USA

Justin Grosslight
History of Science Department, Harvard University, Cambridge, Massachusetts, USA

Derek Smith
Mathematics Department, Lafayette College, Easton, Pennsylvania, USA

In the maximal ring O of octonion integers ("octavians"), let O_m be the set of elements of norm m. Motivated by a question of H. P. Rehm, we ask whether, for a given positive rational integer m, there exists a set $\{\alpha_i\}$ of norm-m octavians such that O_m is partitioned by cosets $O_1\alpha_i$.

For the case when $m = 2$, we characterize all possible intersections $O_1\alpha \cap O_1\alpha'$, where α and α' are any norm-2 octavians, and we show that all norm-2 partitioning sets are equivalent under the automorphisms of the underlying E_8 lattice mod 2. We then provide conditions sufficient to ensure that certain collections of one-sided cosets of O_1 partition O_m, leading to partitioning sets in the new cases $m = 4$, 8, and $2k$, k odd.

Key Words: Cayley number; Cosets; Factorization; Octavian; Octonion; Partition; Rehm; Unit loop.

Mathematics Subject Classification: 17; 17A; 17D; 17A75.

1. INTRODUCTION

In this section, we provide a brief introduction to the properties of the real octonions $\mathbb{O}$ and its maximal order O that are needed in the latter sections of this article. For more information, see the work of Baez (2002), Conway and Smith (2003), Coxeter (1946), and van der Blij and Springer (1959). We follow the notation in Conway and Smith (2003).

Received November 22, 2004; Revised January 6, 2006. Communicated by K. Misra.
Address correspondence to Derek Smith, Mathematics Department, Lafayette College, Easton, PA 18042, USA; Fax: (610) 330-5721; E-mail: smithder@lafayette.edu

The octonions $\mathbb{O}$ form an 8-dimensional noncommutative, nonassociative algebra over $\mathbb{R}$. Each element $\alpha \in \mathbb{O}$ can be expressed in the form

$$\alpha = a_\infty i_\infty + a_0 i_0 + a_1 i_1 + \cdots + a_6 i_6,$$

where we write $i_\infty = 1$ and where the units $i_0, \ldots, i_6$ satisfy

$$i_0^2 = \cdots = i_6^2 = -1$$

and

$$(i_n i_{n+1}) i_{n+3} = i_n (i_{n+1} i_{n+3}) = -1 \qquad \text{(subscripts mod 7)}.$$

A celebrated theorem of Hurwitz states that $\mathbb{R}$, $\mathbb{C}$, Hamilton's quaternions $\mathbb{H}$, and $\mathbb{O}$ are, up to isotopy, the only composition algebras for the Euclidean norm $[x] = \sum x_i^2$ over $\mathbb{R}$, meaning that the composition law $[\alpha][\beta] = [\alpha\beta]$ holds for all α, β.

We shall use $[x, y] = \frac{1}{2}([x+y] - [x] - [y])$ to denote the inner product associated with $[x]$. $\mathbb{O}$ is equipped with an involutionary conjugation map $\alpha \to \bar{\alpha} = 2[\alpha, 1] - \alpha$ such that $\alpha\bar{\alpha} = [\alpha]$. Two other important properties relating the inner product and conjugation to octonion multiplication are listed below, followed by the Moufang laws:

$[\alpha, \gamma\bar{\beta}] = [\alpha\beta, \gamma] = [\beta, \bar{\alpha}\gamma]$	The Braid Laws
$[\alpha\beta, \alpha\gamma] = [\alpha][\beta, \gamma] = [\beta\alpha, \gamma\alpha]$	The Scaling Laws
$\alpha(\beta\gamma)\alpha = (\alpha\beta)(\gamma\alpha)$	The (Bi-)Moufang Law
$\alpha(\beta\gamma) = (\alpha\beta\alpha)(\alpha^{-1}\gamma)$	Left Moufang Law
$(\beta\gamma)\alpha = (\beta\alpha^{-1})(\alpha\gamma\alpha)$	Right Moufang Law.

Apart from the Moufang laws, the most important property concerning the associativity of certain collections of octonions is Artin's diassociativity theorem: Any subalgebra of $\mathbb{O}$ generated by two octonions is associative.

The most obvious ring of integers in $\mathbb{O}$ is the Gravesian ring G, the $\mathbb{Z}$-linear span of $1, i_0, \ldots, i_6$, but the most useful ring is one of the seven isomorphic maximal orders containing G, which we call the *octavians* and denote by O. In terms of the units $1, i_0, \ldots, i_6$, any element $\alpha \in \mathsf{O}$ has coordinates in $\frac{1}{2}\mathbb{Z}$, with those coordinates in $\mathbb{Z}$ corresponding to 1 of 16 "halving-sets" (see Conway and Smith, 2003) of subscripts:

∅	0124	0235	0346	∞450	0561	∞602	∞013
∞0123456	∞356	∞146	∞125	1236	∞234	1345	2456.

O is geometrically similar to the E_8 root lattice, the unique even unimodular lattice in 8 dimensions. The set O_m of octavians of norm $m > 0$ thus has size $240\sigma_3(m)$, where $\sigma_3(m) = \sum_{d|m} d^3$. There are 240 unit octavians; by the composition law, the set of units is closed under multiplication and thus forms a finite subloop $U = \mathsf{O}_1$ of O.

The limited associativity in O causes the details of its arithmetic to be more subtle than those of related associative structures. For instance, the "modified"

Euclidean algorithm that leads to the unique factorization result in Rehm (1993) is necessitated by the fact that ideals in O are trivial (see Allcock, 1999; Lamont, 1963). Also, the action of the loop U on sets of octavians does not behave in a manner akin to group actions since, e.g., it is not true that $U\alpha$ and $U\alpha'$ are either identical or disjoint for octavians α, α'.

In this article, we investigate an important problem posed by Rehm (1993). Namely, we address the *partition problem*: For a given positive integer m, does there exist a set $\{\alpha_1, \ldots, \alpha_{\sigma_3(m)}\}$ of norm-m octavians such that O_m is the disjoint union of all $U\alpha_i$? For $m = 2$, we characterize all possible sets $U\alpha \cap U\alpha'$, where α and α' are any norm-2 octavians, and we show that all norm-2 partitioning sets $\{\alpha_1, \ldots, \alpha_9\}$ are equivalent up to the automorphisms of the underlying E_8 lattice mod 2. We then further extend the results of Rehm by constructing such partitions of O_m for $m = 4$, 8, and $2 \cdot odd$ after providing two general conditions sufficient for a set of octavians to induce a partition.

It will be necessary to consider the octavians modulo 2, so we provide a few facts about O/2O here. The 2^8 classes of O/2O are of three different types: there is one class represented by the 0 vector, 120 classes represented by $\pm$ pairs of unit octavians, and 135 classes represented by orthogonal "frames" of ± 8 octavians of norm 2. The map $O \to O/2O$ sends octavians of odd norm to the unit classes and those of even norm to the remaining classes. For a further discussion of O/2O, see Conway and Smith (2003).

2. PARTITIONING SETS OF LAMONT AND REHM

The following lemma leads to a simple condition ensuring that $U\alpha \cap U\alpha' = \emptyset$ for $\alpha, \alpha' \in O_m$.

Lemma 1. *If $\epsilon\alpha = \epsilon'\alpha'$, where $[\epsilon] = [\epsilon'] = n$ and $[\alpha] = [\alpha'] = m$, then*

$$n[\alpha, \alpha'] = m[\epsilon, \epsilon'].$$

Proof. Taking the inner product of $\epsilon\alpha = \epsilon'\alpha'$ with $\epsilon\alpha'$ yields

$$n[\alpha, \alpha'] = [\epsilon\alpha, \epsilon\alpha'] = [\epsilon'\alpha', \epsilon\alpha'] = m[\epsilon, \epsilon']$$

in view of the scaling laws. □

Since the only possible inner products for a pair ϵ, ϵ' of units are ± 1, $\pm\frac{1}{2}$, 0, we deduce the condition

$$U\alpha \cap U\alpha' = \emptyset \quad \text{when } \alpha, \alpha' \in O_m \text{ and } [\alpha, \alpha'] \notin \{\pm m, \pm m/2, 0\}. \qquad \mathbf{(P)}$$

Thus, norm-m octavians $\alpha_1, \ldots, \alpha_{\sigma_3(m)}$ induce a partition

$$O_m = \bigcup_i U\alpha_i$$

if $[\alpha_i, \alpha_j] \notin \{\pm m, \pm m/2, 0\}$ for $i \neq j$.

Condition (P) applies to octavians whose norms are equal to any positive integer m, providing easy verification that the cases considered by Rehm, when m is odd or equal to 2, are indeed partitions of O_m. For odd m, Rehm (as also in Lamont, 1973) sets $\{\alpha_1, \ldots, \alpha_{\sigma_3(m)}\}$ equal to the set of all octavians of norm m, after selecting one from each $\pm$ pair, that are congruent to 1 mod 2. Then $U\alpha_i \cap U\alpha_j = \emptyset$ follows from (P) since $\alpha_i \equiv \alpha_j (\mathrm{mod}\, 2)$ implies that $\alpha_i - \alpha_j = 2\gamma$ for some $\gamma \in \mathsf{O}$, and so

$$4[\gamma] = [2\gamma] = [\alpha_i - \alpha_j] = [\alpha_i] + [\alpha_j] - 2[\alpha_i, \alpha_j] = 2(m - [\alpha_i, \alpha_j]).$$

Since $[\gamma] \in \mathbb{Z}$ and m is odd, $[\alpha_i, \alpha_j]$ must be an odd integer, and so

$$[\alpha_i, \alpha_j] \in \{\pm m, \pm m/2, 0\} \text{ implies } [\alpha_i, \alpha_j] = \pm m,$$

which is the case just when $\alpha_i = \pm\alpha_j$. We remark that taking the set of all octavians of norm m congruent to any unit, then selecting one from each $\pm$ pair, would produce a partitioning set. We also note that there are other possible partitioning sets for odd m. For $m = 3$, one such set consists of all octavians whose coefficient of i_∞ is equal to 3/2 and which have $\pm 1/2$'s, with an even number of $-1/2$'s, on the remaining three coordinates of a halving set of four coordinates that mentions ∞.

A similar construction of $\{\alpha_1, \ldots, \alpha_{\sigma_3(m)}\}$ for even m is not possible, since octavians of even norm map to mod 2 classes not represented by units. With the help of a computer search and verification, Rehm provided the following example for $m = 2$:

$$\begin{aligned}
\alpha_1 &= 1 + i_0\\
\alpha_2 &= \frac{1}{2}(2 + i_0 + i_3 + i_4 + i_6)\\
\alpha_3 &= \frac{1}{2}(2 + i_0 + i_3 - i_4 + i_6)\\
\alpha_4 &= \frac{1}{2}(2 + i_0 + i_1 + i_5 + i_6)\\
\alpha_5 &= \frac{1}{2}(2 + i_0 - i_1 + i_5 + i_6)\\
\alpha_6 &= \frac{1}{2}(2 + i_0 + i_2 + i_3 + i_5)\\
\alpha_7 &= \frac{1}{2}(2 + i_0 - i_2 + i_3 + i_5)\\
\alpha_8 &= \frac{1}{2}(1 + 2i_0 + i_3 + i_5 + i_6)\\
\alpha_9 &= \frac{1}{2}(1 + 2i_0 - i_3 - i_5 - i_6).
\end{aligned}$$

Using (P), the verification that these α_i induce a partition of O_2 is trivial, since none of the 36 inner products $[\alpha_i, \alpha_j]$ with $i \neq j$ is in $\{\pm 2, \pm 1, 0\}$.

3. EQUIVALENCE OF PARTITIONING SETS FOR $m = 2$

This section is devoted to showing that all partitioning sets for $m = 2$ are equivalent under the automorphisms of the underlying E_8 lattice mod 2. We first prove the following theorem.

Theorem 1. *Norm-2 octavians $\alpha_1, \ldots, \alpha_9$ induce the partition $\mathrm{O}_2 = \bigcup_i U\alpha_i$ if and only if $[\alpha_i, \alpha_j] \notin \{\pm 2, \pm 1, 0\}$ for $i \neq j$.*

We remark that for $m \geq 3$, norm-m octavians α, α' may have $U\alpha \cap U\alpha' = \emptyset$ when $[\alpha, \alpha'] \in \{\pm m, \pm m/2, 0\}$.

Theorem 1 is a consequence of the following lemma, whose proof shows that $U\alpha$ only depends on α mod 2 when α has norm 2.

Lemma 2. *If two norm-2 octavians α, α' have $[\alpha, \alpha'] \in \{\pm 2, \pm 1, 0\}$, then either $U\alpha = U\alpha'$ (in the case that $\alpha \equiv \alpha' \pmod 2$) or $|U\alpha \cap U\alpha'| = 48$.*

Proof. Since $u\alpha = u'\alpha'$ is equivalent by diassociativity to $2u = (u'\alpha')\bar{\alpha}$, we are interested in the units u' for which $(u'\alpha')\bar{\alpha}$ is twice a unit. If $\alpha \equiv \alpha' \pmod 2$, so that $\alpha' = \alpha + 2\gamma$, $\gamma \in \mathrm{O}$, then

$$(u'\alpha')\bar{\alpha} = (u'(\alpha + 2\gamma))\bar{\alpha} = 2(u' + (u'\gamma)\bar{\alpha})$$

has norm 4 and is in $2\mathrm{O}$, so $(u'\alpha')\bar{\alpha}$ is twice a unit for all $u' \in U$. Thus, $U\alpha = U\alpha'$ when $\alpha' \equiv \alpha \pmod 2$.

An enumeration of the several cases, up to $\mathrm{Aut}(\mathrm{O}/2\mathrm{O})$, of pairs of norm-2 octavians α, α' such that $\alpha \not\equiv \alpha' \pmod 2$ and $[\alpha, \alpha'] \in \{\pm 2, \pm 1, 0\}$ reveals that $|U\alpha \cap U\alpha'| = 48$ in each case. We now describe the structure of $U\alpha \cap U\alpha'$ for these pairs. First, we note that the automorphism group of the underlying E_8 lattice (see Conway and Sloane, 1993) is transitive on pairs β, β' of norm 2 vectors with a fixed inner product, as long as $\beta \not\equiv \beta' \pmod 2$. For each such pair, we can always find representatives α, α' of the mod 2 classes $[\beta], [\beta']$ whose inner product is 1.

Since $[\alpha, \alpha'] = 1$, any solution $u\alpha = u'\alpha'$ by a pair (u, u') of units implies

$$\mathrm{Re}(\bar{u}u') = [1, \bar{u}u'] = [u, u'] = \frac{1}{2}$$

by Lemma 1 and a braid law, so that $\bar{u}u'$ is a sixth root of unity. By diassociativity and a Moufang law,

$$u\alpha = u'\alpha' \Longleftrightarrow \alpha = \bar{u}(u'\alpha') = (\bar{u}u'\bar{u})(u\alpha') \Longleftrightarrow (u\overline{u'}u)\alpha = u\alpha'.$$

Applying this conversion again to $(u\overline{u'}u)\alpha = u\alpha'$, we conclude that

$$u\alpha = u'\alpha' \Longleftrightarrow (u\overline{u'}u)\alpha = u\alpha' \Longleftrightarrow -u'\alpha = (u\overline{u'}u)\alpha'.$$

From this we deduce two things. The set S of units u such that there exists u' to satisfy $u\alpha = u'\alpha'$ is equal to the set of u', since if (u, u') is a solution, then so is $(u', -u\overline{u'}u)$. Also, any unit u_0 in S determines two other units, $u_1 = u_0\overline{u_0'}u_0$ and $u_2 = -u_0'$, in S such that $[u_0, u_1] = [u_1, u_2] = [u_2, u_0] = -1/2$.

Letting (u, u') be any solution of $u\alpha = u'\alpha'$, all 48 solutions are obtained as follows. There are 16 solutions generated by letting α vary over the elements of $[\alpha]$. By Lemma 1, each of the sets of 16 corresponding units $\{u\}$ and $\{u'\}$, like $[\alpha]$, forms a frame of ± 8 mutually orthogonal octavians (and their negatives). The set of all 48 solutions are then generated from these 16 by

$$(u, u') \to (u\overline{u'}u, u) \to (-u', u\overline{u'}u).$$

These solutions are distinct since, e.g., a left-hand unit in one frame of 16 has nonintegral inner product with the left-hand units in the other two frames. □

We are now in position to prove the following theorem.

Theorem 2. *All partitioning sets of nine norm-2 octavians are equivalent under the automorphisms of the underlying E_8 lattice* mod 2.

Proof. We employ the even coordinate system for E_8, in which E_8 consists of all vectors $(x_1, \dots, x_8)$ such that $\sum_i x_i$ is even, with either all $x_i \in \mathbb{Z}$ or all $x_i \in \mathbb{Z} + \frac{1}{2}$. With this description, octavians of norm n in O correspond to vectors of norm $2n$ in E_8.

The group $\mathrm{Aut}(E_8)$, of order $192 \cdot 10!$, has a simple description for our purposes: $\mathrm{Aut}(E_8)$ is transitive on the 2160 vectors of norm 4, and upon fixing the vector $v_1 = (2, 0, 0, 0, 0, 0, 0, 0)$ the remaining $2^6 \cdot 7!$ symmetries are precisely the even sign changes and permutations of the final 7 coordinates. (The 16 vectors congruent to v_1 mod 2 are those of the form $(\pm 2)^1 0^7$, so fixing only the class $[v_1]$ leaves the $2^7 \cdot 8!$ even sign changes and permutations of all eight coordinates.)

Since by Theorem 1 we seek a set $\{v_1, \dots, v_9\}$ of norm 4 vectors in E_8 such that $[v_i, v_j]$ is odd for $i \neq j$, the remaining vectors $v_2, \dots, v_9$ must all be of the form $(\pm\frac{3}{2})^1(\pm\frac{1}{2})^7$ with an odd number of minus signs. There are 64 classes of such vectors mod 2, each class containing the vectors (and their negatives) obtained from a norm-2 vector of the form $(\pm\frac{1}{2})^8$ with an even number of minus signs by replacing a single $+\frac{1}{2}$ with $-\frac{3}{2}$ or a single $-\frac{1}{2}$ with $+\frac{3}{2}$. Thus, we may choose representatives $v_2, \dots, v_9$ to have $\frac{3}{2}$ in their first coordinate and an odd number of $-\frac{1}{2}$'s in the remaining seven.

Using the symmetries of E_8 remaining after fixing v_1, it is easy to check that the following norm-4 vectors in E_8 may be taken as representatives of mod 2 equivalence classes for $v_1, \dots, v_9$:

$$v_1 = (2, 0, 0, 0, 0, 0, 0, 0)$$
$$v_2 = \frac{1}{2}(3, -1, -1, -1, -1, -1, -1, -1)$$
$$v_3 = \frac{1}{2}(3, -1, 1, 1, 1, 1, 1, 1)$$

$$v_4 = \frac{1}{2}(3, 1, -1, 1, 1, 1, 1, 1)$$

$$v_5 = \frac{1}{2}(3, 1, 1, -1, 1, 1, 1, 1)$$

$$v_6 = \frac{1}{2}(3, 1, 1, 1, -1, 1, 1, 1)$$

$$v_7 = \frac{1}{2}(3, 1, 1, 1, 1, -1, 1, 1)$$

$$v_8 = \frac{1}{2}(3, 1, 1, 1, 1, 1, -1, 1)$$

$$v_9 = \frac{1}{2}(3, 1, 1, 1, 1, 1, 1, -1).$$ □

4. PARTITIONING SETS FOR $m = 4$, 8, AND $2k$, k ODD

In this final section, we provide partitioning sets for O_m in several new cases.

Theorem 3. *Partitioning sets for* O_{2k}, *k exist.*

Proof. Let $m = 2k$, k odd, let $\{\alpha_i\}$ be the partitioning set of norm-k vectors from the Section 2, and let $\{\beta_i\}$ be any norm-2 partitioning set from Section 3. Then $\{\alpha_i\beta_j\}$ is a norm-$2k$ partitioning set.

To see this, suppose that there exist units u, u' such that $u(\alpha\beta) = u'(\alpha'\beta')$ for $\alpha, \alpha' \in \{\alpha_i\}$ and $\beta, \beta' \in \{\beta_i\}$. Then since $\alpha \equiv \alpha' \pmod 2$ we have

$$2k[u, u'] = [\alpha'\beta', \alpha\beta] = [(\alpha + 2\gamma)\beta', \alpha\beta] = k[\beta', \beta] + 2[\gamma\beta', \alpha\beta],$$

for some $\gamma \in \mathrm{O}$. But $2k[u, u']$ and $2[\gamma\beta', \alpha\beta]$ are integral, while $k[\beta', \beta]$ is nonintegral if $\beta' \neq \beta$. If $\beta' = \beta$, the first equality above reduces to $k[u, u'] = [\alpha, \alpha']$, which is ruled out by the argument for odd-norm elements in Section 2. □

Theorem 4. *Partitioning sets for* O_4 *and* O_8 *exist.*

Proof. Let $m = 2^n$, $n \geq 2$, for which the following refinement of (P) holds:

$$U\alpha \cap U\alpha' = \emptyset \quad \text{when } \alpha, \alpha' \in \mathrm{O}_{2^n}, \qquad \alpha \equiv \alpha' \pmod 2,$$
$$\text{and} \qquad [\alpha, \alpha'] \notin \{\pm 2^n, 0\}. \qquad (\mathbf{P}')$$

To see this, first note that if $\alpha = 2^a\tau$ and $\alpha = 2^{a'}\tau'$ for primitive octavians τ, τ', the sets $U\alpha$ and $U\alpha'$ are disjoint if $a \neq a'$, and if $a = a'$ they are disjoint if and only if $U\tau$ and $U\tau'$ are disjoint, so we need only consider the case when α, α' are primitive. Suppose that there exist units u, u' such that $u\alpha = u'\alpha'$, which holds if and only if $(u\alpha)\overline{\alpha'} \in 2^n\mathrm{O}$. Since

$$(u\alpha)\overline{\alpha'} = (u(\alpha' + 2\tau))\overline{\alpha'} = 2^n u + 2(u\gamma)\overline{\alpha'},$$

for some $\gamma \in \mathcal{O}$, $(u\alpha)\overline{\alpha'} \in 2^n\mathcal{O}$ if and only if $(u\gamma)\overline{\alpha'} \in 2^{n-1}\mathcal{O}$. Since α is primitive, by Conway and Smith (2003, Chapter 9, Theorem 6), we have $(u\gamma)\overline{\alpha'} \notin 2^{n-1}\mathcal{O}$ unless 2^{n-1} divides $[\gamma]$. But

$$[\gamma] = [(1/2)(\alpha - \alpha')] = (1/2)(2^n - [\alpha, \alpha']),$$

so that 2^{n-1} does not divide $[\gamma]$ if $[\alpha, \alpha'] = \pm 2^{n-1}$.

This leads to a norm-2^n generating set in the following way. Of the $240\sigma_3(2^n)$ octavians of norm 2^n, exactly $240((2^{n-1})^3 + (2^n)^3) = 240 \cdot 9 \cdot 2^{3n-3}$ are primitive. Let $\{\beta_i\}$ be a norm-2 partitioning set of nine octavians. For each β_i, let $\{\alpha_{j,\beta_i}\}$ be a set of 2^{3n-3} primitive octavians of norm 2^n, each congruent to $\beta_i \bmod 2$ and no two of which are orthogonal or negatives of each other. Then take the union of this set of primitive norm-2^n elements with imprimitive norm-2^n elements of the form $2^k\tau$, where $n/2 \geq k > 0$ and τ represents a primitive octavian of norm 2^{n-2k} generated in the same manner as the α_{j,β_i}.

We now describe partitioning sets for $m = 4$ and $m = 8$ using the even coordinate system for E_8, so the corresponding vectors have norms 8 and 16, respectively. For $m = 4$, an example of 8 primitive, pairwise nonorthogonal vectors of norm 8 that are congruent mod 2 are all permutations of $\frac{5}{2}^1\frac{1}{2}^7$. A partitioning set is constructed from the images of these vectors under 9 automorphisms of E_8 mapping $-\frac{3}{2}^1\frac{1}{2}^7$, say, to the v_i at the end of Section 3, along with any single vector of norm 8 in $2E_8$. For $m = 8$, an example of 64 primitive, pairwise nonorthogonal vectors $(x_1, \ldots, x_8)$ of norm 16 that are congruent mod 2 are those with $x_1 = 3$ and $x_2, \ldots, x_8$ equal to ± 1, with an even number of $+1$s. Then a full partitioning set is created by images of of these vectors under nine automorphisms of E_8 mapping $(2, 0, \ldots, 0)$ to the v_i, along with the nine vectors $2v_i$. □

We are not aware of any norm-2^n partitioning sets for $n \geq 4$.

ACKNOWLEDGMENT

This article was partially supported by NSF grant DMS-9912325.

REFERENCES

Allcock, D. (1999). Ideals in the integral octaves. *J. Algebra* 220:396–400.

Baez, J. (2002). The octonions. *Bull. Amer. Math. Soc.* 39:145–205.

Conway, J. H., Sloane, N. J. A. (1993). *Sphere Packings, Lattices, and Groups*. 2nd ed. Springer-Verlag.

Conway, J. H., Smith, D. A. (2003). *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry*. AK Peters.

Coxeter, H. S. M. (1946). Integral Cayley numbers. *Duke Math. J.* 13:561–578.

Lamont, P. J. C. (1963). Ideals in Cayley's algebra. *Indagationes Mathematicae* 25:394–400.

Lamont, P. J. C. (1973). Factorization and arithmetic functions for orders in composition algebras. *Glasgow Math. J.* 14:86–95.

Rehm, H. P. (1993). Prime factorization of integral Cayley octaves. *Annales de la Faculté des Sciences de Toulouse* 2(2):271–289.

van der Blij, F., Springer, T. A. (1959). The arithmetics of octaves and of the group G_2. *Nederl. Akad. Wetensch. Indag. Math.* 21:406–418.